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Quantum discrete phase space dynamics and its continuous limit

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Abstract. The main aspects of a discrete phase space formalism are presented and the discrete dynamical bracket, suitable for the description of time evolution in finite-dimensional spaces, is discussed. A set of operator bases is defined in such a way that the Weyl–Wigner formalism is shown to be obtained as a limiting case. In the same form, the Moyal bracket is shown to be the limiting case of the discrete dynamical bracket. The dynamics in quantum discrete phase spaces is shown not to be attained from discretization of the continuous case.

1. Introduction

The non-relativistic quantum phase space picture of quantum mechanics has deserved attention for a long time as an interesting alternate framework for studying quantum systems [1–8]. In its foundations, this approach basically deals with a particular correspondence between abstract operators and certain functions of a pair of continuous *c*-variables, which represent them in the quantum phase space, in such a form that the whole physical content is preserved. It has been shown that such a correspondence can be implemented through a particular mapping that allows one to represent the operators as functions, using the concept of a pseudo-distribution, such as the Wigner function associated with the density operator [2], introduced in order to account for the state of the physical system. In this connection, the well known Weyl–Wigner mapping has been clearly established and discussed in relation to degrees of freedom with classical counterparts and, in addition, it has also been shown that, in the phase space description, the time evolution of a system is governed by the action of the Moyal bracket [3] over the Wigner function. From an algebraic point of view, the Moyal bracket can be seen as a deformation of the classical Poisson bracket [9], and plays an essential role in the dynamics on continuous quantum phase spaces.

At the same time, it has also been verified that it is possible to establish the foundations for quantum *discrete* phase space descriptions in a way closely paralleling the continuous formulation. In this case, a proposed association technique between operators and functions of integers is strongly based on the Schwinger prescription for constructing bases in operator spaces when one deals with degrees of freedom characterized by finite-dimensional state spaces [10]. The Schwinger basis is known to have a factorization property, which allows for a separation of degrees of freedom [10], each one being described by a prime number of states. These operator bases, in turn, are constructed out of a family of products of cyclic unitary operators which obey a particular commutation relation sometimes known as Weyl–Schwinger algebra and, due to this inherent structure, the Schwinger operator bases elements can be shown to fulfil the more general Fairlie–Fletcher–Zachos sine algebra [11, 12]. It is immediately seen that the association of the discrete labels of Schwinger's unitary operators with the points of a finite-dimensional lattice can indeed define such a phase space, as has been proposed in the past [13–15]. In what refers to the state of the finite-dimensional system, its phase space representative is a discrete Wigner function associated with the density operator [13, 16] and has also been discussed in the context of prime factorization in [17, 18]. In spite of the formalism developed for describing finite-dimensional degrees of freedom is exact, it can also be used as an approximation technique for continuous quantum systems as rigorously discussed in [19].

Although the operator basis proposed by Schwinger provides for a natural starting point for phase space descriptions, another one has been proposed in the past which is the double Fourier transform of that of Schwinger, and that explicitly implements a suitable mod N symmetry [13, 20–22], keeping the desirable property of prime factorization. As in the continuous case, the description based on these operator bases can describe the time evolution of a selected state of the physical system of interest in the corresponding discrete phase space, either for the independent degrees of freedom, sieved by the Schwinger factorization, or for the full discrete phase space as a whole. In this connection, it was established that the discrete mapped version of the Liouville operator defines a quantum dynamical bracket which fully describes the time evolution in the discrete phase space, and is associated with transformations that preserve the presymplectic structure [23], with a role similar to that of the Moyal bracket in the continuous case.

In this paper we want to address two questions. First, since it is known that when the state space dimension goes to infinity ($N \to \infty$), the algebra of the Schwinger unitary operators is deeply related to that of the quantum canonical pair $\{\hat{q}, \hat{p}\}$, it is to be expected that the mapping kernels must contain a limiting element which describes the continuous degree of freedom. To discuss this question we propose here a new operator basis which is a modification of the operator basis elements mentioned above, in such a form as to explicitly symmetrize the discrete phase space labels range. The properties of this operator basis are obtained, and are seen to be the discrete equivalent of the usual continuous case, i.e. the Weyl–Wigner approach. Furthermore, we show that the limiting element, obtained through a convenient procedure, is exactly the Weyl–Wigner continuous mapping kernel. The symmetrization of the new basis also proves itself very important in connection with our second question: what is the behaviour of the discrete dynamical bracket for $N \to \infty$? In this connection, we will show here that the Moyal bracket [3] emerges as the continuous limit of the discrete dynamical bracket. Furthermore, as a byproduct of this deduction, we point to the inherent difficulties in obtaining the discrete dynamical bracket from that of Moyal, due to particular features of the discrete expression that could only be recovered in a very artificial way.

Our paper is organized as follows. In section 2 we present a brief review of the continuous Weyl–Wigner approach, and the discrete phase space formalism is presented in section 3. The continuous case is reattained, as the $N \to \infty$ limit, in section 4, and the concluding remarks are presented in section 5.

2. Brief review of the Weyl–Wigner formalism

Since we will be interested in comparing quantum continuous and discrete phase space mapping techniques, let us briefly recall, in what follows, some basic results associated with one degree of freedom.

In the continuous case one takes advantage of the resolution of unity in both momentum and coordinate representations in order to directly verify that an operator $\hat{A}$ can be written as [4, 24]

$$\hat{A} = \frac{1}{2\pi\hbar}\int \mathrm{d}p\,\mathrm{d}q\, a(p,q)\Delta(p,q) \tag{1}$$

with

$$a(p,q) = \int \mathrm{d}u \exp\left[\frac{\mathrm{i}}{\hbar}qu\right]\langle p+\tfrac{1}{2}u|\hat{A}|p-\tfrac{1}{2}u\rangle \tag{2}$$

and

$$\Delta(p,q) = \int \mathrm{d}v \exp\left[\frac{\mathrm{i}}{\hbar}pv\right]|q+\tfrac{1}{2}v\rangle\langle q-\tfrac{1}{2}v|. \tag{3}$$

The function $a(p,q)$ is called the Weyl transform of the operator $\hat{A}$ with respect to the momentum and coordinate operators $\hat{P}$ and $\hat{Q}$. At the same time, one recognizes $\Delta(p,q)$ as the continuous elements of an operator basis. Therefore, the Weyl transform of an operator $\hat{A}$ is then readily understood as the coefficient of its decomposition in the operator basis $\Delta(p,q)$. Also, due to symmetry arguments, one can interchange $q \to p$ and $p \to -q$ without any loss of physical content in the mapping.

The Weyl transform is also obtained through the trace operation

$$a(p,q) = \mathrm{Tr}[\Delta(p,q)\hat{A}] \tag{4}$$

and the operator basis elements have the following basic properties:

$$\mathrm{Tr}[\Delta(p,q)] = 1 \tag{5}$$

$$\mathrm{Tr}[\Delta(p,q)\Delta(p',q')] = 2\pi\hbar\delta(p-p')\delta(q-q') \tag{6}$$

$$\mathrm{Tr}[\Delta(p,q)\Delta(p',q')\Delta(p'',q'')] = 2^2\exp\left[\frac{2\mathrm{i}}{\hbar}\left[p(q''-q')+p'(q-q'')+p''(q'-q)\right]\right]. \tag{7}$$

It is not difficult to verify from equations (1), (5) and (6) that

$$\mathrm{Tr}[\hat{A}] = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\mathrm{d}p\,\mathrm{d}q\, a(p,q) \tag{8}$$

and

$$\mathrm{Tr}[\hat{F}\hat{G}] = \frac{1}{2\pi\hbar}\int \mathrm{d}p\,\mathrm{d}q\, f(p,q)g(p,q). \tag{9}$$

From equation (7) one obtains the mapped expression for the product of two operators, namely

$$(\hat{A}\hat{B})(p,q) = \frac{2^2}{(2\pi\hbar)^2}\int \mathrm{d}p''\,\mathrm{d}q''\,\mathrm{d}p'\,\mathrm{d}q'\, a(p',q')b(p'',q'') \times\exp\left[\frac{2\mathrm{i}}{\hbar}\left[(q'-q)(p''-p)-(q''-q)(p'-p)\right]\right] \tag{10}$$

where $(\hat{X})(p,q)$ stands for the phase space mapped representative of an operator $\hat{X}$. After some algebraic manipulation equation (10) can be put in the form

$$(\hat{A}\hat{B})(p,q) = \exp\left[\frac{\hbar}{2\mathrm{i}}\left(\frac{\partial}{\partial p}^{a}\frac{\partial}{\partial q}^{b} - \frac{\partial}{\partial q}^{a}\frac{\partial}{\partial p}^{b}\right)\right]a(p,q)b(p,q) \tag{11}$$

where the indices indicate over which function the derivatives act and, of course, the exponential stands for its power expansion and, from an algebraic point of view, equation (11) can also be understood as a composition of phase space functions with an associative star product [25–27]. From this result one easily obtains the Weyl transformations of the commutator and anticommutator of $\hat{A}$ and $\hat{B}$,

$$([\hat{A},\hat{B}])(p,q) = 2\mathrm{i}\sin\left[\frac{\hbar}{2\mathrm{i}}\left(\frac{\partial}{\partial p}^{a}\frac{\partial}{\partial q}^{b} - \frac{\partial}{\partial q}^{a}\frac{\partial}{\partial p}^{b}\right)\right]a(p,q)b(p,q) \tag{12}$$

$$(\{\hat{A},\hat{B}\})(p,q) = 2\cos\left[\frac{\hbar}{2\mathrm{i}}\left(\frac{\partial}{\partial p}^{a}\frac{\partial}{\partial q}^{b} - \frac{\partial}{\partial q}^{a}\frac{\partial}{\partial p}^{b}\right)\right]a(p,q)b(p,q). \tag{13}$$

Expression (12) is known as the Moyal bracket of two operators [3], and it stands for the representative of the commutator in terms of the two commuting classical variables q and p. It can immediately be seen that the anticommutator series starts with the product of the Weyl transforms of the operators $\hat{A}$ and $\hat{B}$, while the commutator series starts with $i\hbar$ times the Poisson bracket of $\hat{A}$ and $\hat{B}$.

2.1. Dynamics

Considering now the density operator of a physical system associated, for simplicity, to a pure state,

$$\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)|. \tag{14}$$

Its Weyl transform, in the wavefunction notation $\psi(q;t) \equiv \langle q|\psi(t)\rangle$, reads

$$\rho_w(p,q;t) = \int_{-\infty}^{\infty} \mathrm{d}v \exp\left(\frac{\mathrm{i}}{\hbar}pv\right)\psi(q - \tfrac{1}{2}v;t)\psi^*(q + \tfrac{1}{2}v;t). \tag{15}$$

The function $\rho_w(p,q;t)$ is called the Wigner function and extensive discussions of its interpretation and properties can be found in [2–8].

One can see that the expectation value of an operator can be written as

$$\bar{A}(t) = \mathrm{Tr}[\hat{\rho}(t)\,\hat{A}] \tag{16}$$

and, from equation (9) it follows that

$$\bar{A}(t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \mathrm{d}p\,\mathrm{d}q\, a(p,q)\,\rho_w(p,q;t). \tag{17}$$

In what refers to the dynamics it is well known that the time evolution of a quantum system can be described by the von Neumann–Liouville equation for the density operator, namely

$$\frac{\partial}{\partial t}\hat{\rho}(t) = -\frac{\mathrm{i}}{\hbar}[\hat{H},\hat{\rho}(t)]. \tag{18}$$

With the help of equation (12), this equation can be directly mapped onto its equivalent in the continuous quantum phase space,

$$\frac{\partial}{\partial t}\rho_w(p,q;t) = \frac{2}{\hbar}\sin\left[\frac{\hbar}{2\mathrm{i}}\left(\frac{\partial}{\partial p}^{h}\frac{\partial}{\partial q}^{\rho} - \frac{\partial}{\partial q}^{h}\frac{\partial}{\partial p}^{\rho}\right)\right]h(p,q)\rho_w(p,q) \tag{19}$$

where $h(p,q)$ is the mapped expression of the Hamiltonian. Now, this equation describes the dynamics of a quantum system in the Weyl–Wigner formalism and clearly stresses the important role played by the Moyal bracket in this quantum phase space description of the time evolution of the system state. At the same time, the Moyal bracket is at the root of the limit procedure which allows us—concomitantly with the classical limit of the Wigner function—to recover the classical Liouville dynamical equation governing the time evolution in classical phase spaces [24]. Although of great interest, we will not concern ourselves here with the wealthy field of semiclassical approximations.

3. Discrete phase space formalism

In the same spirit as that presented in the previous section, we now can look for a discrete phase space associated with a quantum finite-dimensional degree of freedom without a classical counterpart. In a first approach, we can write (hereafter we will be considering $\hbar = 1$)

$$O(m, n) = \mathrm{Tr}[B^{\dagger}(m, n)\hat{O}] \tag{20}$$

where the operators $B(m, n)$ are elements of any given complete orthonormal operator basis. Schwinger [10] proposed a basis with elements $\{\hat{S}(m, n)\}$ defined as

$$\hat{S}(m, n) = \frac{U^m V^n}{\sqrt{N}} \exp\left[\frac{\mathrm{i}\pi}{N} mn\right] \qquad m, n = 0, 1, 2, \ldots N - 1 \tag{21}$$

which is an orthonormal and complete basis in the operator space associated with the system of interest.

Notwithstanding the conceptual importance of the formal approach to the finite-dimensional operator bases as presented above, a new basis can be introduced which takes full benefit of the quantum discrete canonical-like symmetry of the phase space [10]. To this end we explicitly consider basis elements which implement this symmetry, namely

$$\hat{G}(m, n) = \sum_{j,l=-h}^{h} \hat{T}(m, n; j, l) \qquad h = \tfrac{1}{2}(N - 1) \tag{22}$$

$$\hat{T}(m, n; j, l) = \frac{U^j V^l}{N} \exp\left[\frac{\mathrm{i}\pi}{N} jl\right] \exp\left[-\frac{2\pi \mathrm{i}}{N}(mj + nl)\right] \exp\left[\mathrm{i}\pi\phi(j + h, l + h; N)\right] \tag{23}$$

where

$$\phi(j, l; N) = N I_j^N I_l^N - j I_l^N - l I_j^N \tag{24}$$

and

$$I_k^N = \left[\frac{k}{N}\right] \tag{25}$$

takes the integral part of k with respect to N. In this way, this phase is introduced in this simple form just to carry out all the mod N arithmetic that is involved in the phase space mapping calculations as discussed in [20]. The exponential of the modular phase $\exp[\mathrm{i}\pi\phi(j + h, l + h; N)]$ is easily seen to be equal to 1 when $\{j, l\}$ both lie in the interval $[-h, h]$. On the other hand, the presence of this phase ensures a symmetry in the $\hat{T}(m, n; j, l)$ factors so that

$$\hat{T}(m, n; j, l) = \hat{T}(m, n; j(\mathrm{mod}\, N), l(\mathrm{mod}\, N)) \qquad j, l \in Z. \tag{26}$$

This feature will prove itself very useful when discussing the mapping of products of operators.

The basis elements we propose in equation (22) are in fact a suitable modification of the basis elements presented in [20]. Furthermore, the basis elements presented in [20] have all the properties collected in [22], and all the results found there can be applied here by just taking care of the symmetric summation interval and its consequence on the modular phase ϕ. However, the basis elements of equation (22) have an additional very important property of being Hermitian, what can be easily proved.

The basic properties follow:

$$\mathrm{Tr}\left[\hat{G}(m,n)\right] = 1; \tag{27}$$

$$\mathrm{Tr}\left[\hat{G}^{\dagger}(m,n)\hat{G}(r,s)\right] = N\delta^{[N]}_{m,r}\delta^{[N]}_{n,s}; \tag{28}$$

$$\mathrm{Tr}\left[\hat{G}^{\dagger}(m,n)\hat{G}(u,v)\hat{G}(r,s)\right] = \frac{1}{N^2}\sum_{a,b,c,d=-h}^{h} \exp\left[\frac{\mathrm{i}\pi}{N}(bc-ad)\right]\exp\left[\mathrm{i}\pi\Phi(a,b,c;d;N)\right]$$
$$\times \exp\left[\frac{2\pi\mathrm{i}}{N}\left[a(m-u)+b(n-v)+c(m-r)+d(n-s)\right]\right] \tag{29}$$

where

$$\Phi(a,b,c,d;N) = -\phi(a+c+h,b+d+h;N). \tag{30}$$

Equations (27) and (28) are straightforwardly obtained and, due to its importance, equation (29) is obtained in the appendix, where the role played by the phase ϕ becomes clear. The trace of an arbitrary number of basis elements can be found in [22]. It is worth noting that these expressions are the discrete counterparts of equations (5)–(7), respectively.

Since the operator basis elements are Hermitian, it can be directly seen that Hermitian operators have real phase space representative functions. Furthermore, the mapping of the product of two operators is given by

$$(\hat{O}_1\hat{O}_2)(m,n) = \mathrm{Tr}[\hat{G}^{\dagger}(m,n)\hat{O}_1\hat{O}_2] \tag{31}$$

$$(\hat{O}_1\hat{O}_2)(m,n) = \frac{1}{N^2}\sum_{u,v,r,s=-h}^{h} O_1(u,v)O_2(r,s)\,\mathrm{Tr}[\hat{G}^{\dagger}(m,n)\hat{G}(u,v)\hat{G}(r,s)] \tag{32}$$

$$(\hat{O}_1\hat{O}_2)(m,n) = \frac{1}{N^4}\sum_{u,v,r,s=-h}^{h}\ \sum_{a,b,c,d=-h}^{h} O_1(u,v)O_2(r,s)\exp\left[\frac{\mathrm{i}\pi}{N}(bc-ad)\right]$$
$$\times \exp\left[\mathrm{i}\pi\Phi(a,b,c,d;N)\right]$$
$$\times \exp\left[\frac{2\pi\mathrm{i}}{N}\left[a(m-u)+b(n-v)+c(m-r)+d(n-s)\right]\right]. \tag{33}$$

The trace of an operator follows directly,

$$\mathrm{Tr}[\hat{O}_1] = \frac{1}{N}\sum_{m,n=-h}^{h} O_1(m,n) \tag{34}$$

while the trace of a product of two operators can be obtained using the Hermiticity of the basis elements and equation (28),

$$\mathrm{Tr}[\hat{O}_1\hat{O}_2] = \frac{1}{N^2}\sum_{m,n=-h}^{h}\ \sum_{j,l=-h}^{h} O_1(m,n)O_2(j,l)\,\mathrm{Tr}[\hat{G}^{\dagger}(m,n)\hat{G}(j,l)] \tag{35}$$

$$= \frac{1}{N}\sum_{m,n=-h}^{h} O_1(m,n)O_2(m,n) \tag{36}$$

which is the discrete analogue to equation (9).

Once we have the discrete phase space mapped expression for the product of two operators, it is straightforward to obtain the mapped expressions for the commutator and anticommutator of two operators,

$$[\hat{O}_1, \hat{O}_2](m,n) = \frac{2\mathrm{i}}{N^4} \sum_{u,v,r,s=-h}^{h} \sum_{a,b,c,d=-h}^{h} O_1(u,v) O_2(r,s) \times \exp[\mathrm{i}\pi \Phi(a,b,c,d;N)] \, \Gamma(m,n,u,v,r,s,a,b,c,d;N) \tag{37}$$

and

$$\{\hat{O}_1, \hat{O}_2\}(m,n) = \frac{2}{N^4} \sum_{u,v,r,s=-h}^{h} \sum_{a,b,c,d=-h}^{h} O_1(u,v) O_2(r,s) \times \exp[\mathrm{i}\pi \Phi(a,b,c,d;N)] \, \Gamma_2(m,n,u,v,r,s,a,b,c,d;N) \tag{38}$$

respectively, where

$$\Gamma(m,n,u,v,r,s,a,b,c,d;N) = \sin\left[\frac{\pi}{N}(bc-ad)\right] \times \exp\left[\frac{2\pi\mathrm{i}}{N}[a(m-u)+b(n-v)+c(m-r)+d(n-s)]\right] \tag{39}$$

and

$$\Gamma_2(m,n,u,v,r,s,a,b,c,d;N) = \cos\left[\frac{\pi}{N}(bc-ad)\right] \times \exp\left[\frac{2\pi\mathrm{i}}{N}[a(m-u)+b(n-v)+c(m-r)+d(n-s)]\right]. \tag{40}$$

Equations (37) and (38) are to be compared directly with their continuous counterparts, namely equations (12) and (13).

3.1. Dynamics in the discrete phase space

As before, the time evolution of a system is now described through the action of a dynamical bracket on a density operator defined in a finite-dimensional space (for a pure state for simplicity),

$$\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)| \tag{41}$$

whose Weyl representative is still called the Wigner function

$$\rho_w(m,n;t) = \mathrm{Tr}[\hat{G}^\dagger(m,n)\hat{\rho}(t)]. \tag{42}$$

As in the continuous case the expectation value of an operator can be written as

$$\bar{A}(t) = \mathrm{Tr}[\hat{\rho}(t)\hat{A}] \tag{43}$$

which, upon using (36), gives

$$\bar{A}(t) = \frac{1}{N} \sum_{u,v=-h}^{h} a(u,v)\rho_w(u,v;t). \tag{44}$$

Again, the density operator obeys the von Neumann–Liouville equation,

$$\frac{\partial}{\partial t}\hat{\rho}(t) = -\mathrm{i}[\hat{H}, \hat{\rho}(t)] \tag{45}$$

which can be mapped onto the discrete phase space if we use expression (37)

$$\frac{\partial}{\partial t}\rho_w(m,n;t) = \frac{2\mathrm{i}}{N^4}\sum_{u,v,r,s,a,b,c,d=-h}^{h} h(u,v)\rho_w(r,s;t) \times \exp\left[\mathrm{i}\pi\Phi(a,b,c,d;N)\right]\Gamma(m,n,u,v,r,s,a,b,c,d;N) \quad (46)$$

where $h(u,v)$ stands for the mapped expression of the Hamiltonian. This equation can be rewritten as

$$\frac{\partial}{\partial t}\rho_w(m,n;t) = \sum_{r,s=-h}^{h} \mathcal{L}(m,n,r,s;N)\rho_w(r,s;t) \quad (47)$$

where

$$\mathcal{L}(m,n,r,s;N) = \frac{2\mathrm{i}}{N^4}\sum_{u,v=-h}^{h}\sum_{a,b,c,d=-h}^{h} h(u,v) \times \exp\left[\mathrm{i}\pi\Phi(a,b,c,d;N)\right]\Gamma(m,n,u,v,r,s,a,b,c,d;N) \quad (48)$$

is now identified as the discrete mapped expression of the Liouvillian of the system. Equation (47), the discrete mapped von Neumann–Liouville equation, governs the time evolution of the Wigner function in the discrete phase space and stresses the importance of the mapped expression of the commutator embodied in the Liouvillian.

3.2. The time evolution of the Wigner function

To clarify this last point, let us consider the simple case of time-independent Hamiltonians, and write, as usual,

$$\hat{\rho}(t) = \hat{K}(t,t_0)\hat{\rho}(t_0)\hat{K}^{\dagger}(t,t_0) \quad (49)$$

where $\hat{K}(t,t_0)$ is the time evolution operator, namely,

$$\hat{K}(t,t_0) = \exp\left[-\frac{\mathrm{i}}{\hbar}H(t-t_0)\right]. \quad (50)$$

The corresponding discrete phase space mapped expression reads

$$\rho_w(u,v;t) = \mathrm{Tr}\left[\hat{G}^{\dagger}(u,v)\hat{K}(t,t_0)\hat{\rho}(t_0)\hat{K}^{\dagger}(t,t_0)\right] \quad (51)$$

which, upon using the standard decomposition

$$\hat{\rho}(t_0) = \sum_{r,s}\hat{G}(r,s)\rho_w(r,s;t_0) \quad (52)$$

can be written in a general form as

$$\rho_w(u,v;t) = \sum_{r,s}\mathcal{P}(u,v;t\,|r,s;t_0)\rho_w(r,s;t_0) \quad (53)$$

where

$$\mathcal{P}(u,v;t\,|r,s;t_0) = \mathrm{Tr}\left[\hat{G}^{\dagger}(u,v)\hat{K}(t,t_0)\hat{G}(r,s)\hat{K}^{\dagger}(t,t_0)\right]. \quad (54)$$

This last expression is the mapped propagator of the Wigner function in the discrete phase space.

Now, recalling that

$$\hat{\rho}(t) = \hat{\rho}(t_0) - \frac{\mathrm{i}\,(t-t_0)}{\hbar}\left[H, \hat{\rho}(t_0)\right] + \frac{\mathrm{i}^2\,(t-t_0)^2}{2!\hbar^2}\left[H, \left[H, \hat{\rho}(t_0)\right]\right] - \cdots \tag{55}$$

we see that its mapped expression is

$$\rho_w(u, v; t) = \sum_{r,s}\left\{\delta^{[N]}_{r,u}\delta^{[N]}_{s,v} - \frac{\mathrm{i}\,(t-t_0)}{\hbar}\mathcal{L}(u, v, r, s; N) + \frac{\mathrm{i}^2\,(t-t_0)^2}{2!\hbar^2}\sum_{x,y}\mathcal{L}(u, v, x, y; N)\,\mathcal{L}(x, y, r, s; N) - \cdots\right\}\rho_w(r, s; t_0). \tag{56}$$

This series is a solution to equation (47) [28], and by a direct comparison we identify

$$\mathcal{P}(u, v; t\,|r, s; t_0) = \delta^{[N]}_{r,u}\delta^{[N]}_{s,v} - \frac{\mathrm{i}\,(t-t_0)}{\hbar}\mathcal{L}(u, v, r, s; N) + \frac{\mathrm{i}^2\,(t-t_0)^2}{2!\hbar^2}\sum_{x,y}\mathcal{L}(u, v, x, y; N)\,\mathcal{L}(x, y, r, s; N) - \cdots. \tag{57}$$

This expression always associates, through the repeated action of the time-independent Liouvillian, the Wigner function of integers (r, s) at t_0 with a Wigner function of integers (u, v) at time t, while the phase space grid is kept constant in time. For a more general time dependence of the Hamiltonian, see [28].

As an example of the time evolution procedure let us consider the simple case of the motion of a general magnetic moment in a constant magnetic field. Since the Hamiltonian is, by a convenient choice of axis, simply

$$H = \lambda S_z B \tag{58}$$

then the Liouvillian is given by [28]

$$\mathcal{L}(u, v, r, s; N) = \frac{-\lambda}{N^2}\sum_{c,d,b\neq 0}\exp\left\{\frac{2\pi\mathrm{i}}{N}\left[bv + c(u-r) + d(v-s)\right]\right\}\sin\left[\frac{\pi}{N}bc\right] \times \frac{\cos(\pi b)}{\sin(\pi b/N)}\exp\left[\mathrm{i}\pi\Phi(0, b, c, d; N)\right]. \tag{59}$$

If we choose an eigenstate of the angular momentum for the initial state, the corresponding Wigner function will simply read

$$\rho_w(r, s; t_0) = \delta^{[N]}_{s,s_0}$$

and therefore,

$$\sum_{r,s}\mathcal{L}(u, v, r, s; N)\,\rho_w(r, s; t_0) = \sum_{r,s}\mathcal{L}(u, v, r, s; N)\,\delta^{[N]}_{s,s_0}.$$

Since the summation in r gives $\delta^{[N]}_{c,0}$ and the one in s is trivial, we end up with

$$\sum_{c,d,b\neq 0}\exp\left\{\frac{2\pi\mathrm{i}}{N}\left[bv + cu + d(v-s_0)\right] + \mathrm{i}\pi\Phi(0, b, c, d; N)\right\} \times \frac{\sin(\pi bc/N)\cos(\pi b)}{\sin(\pi b/N)}\delta^{[N]}_{c,0} = 0 \tag{60}$$

which guarantees that all terms but the first in equation (56) vanish (equivalently the Wigner propagator is given only by $\delta^{[N]}_{r,u}\delta^{[N]}_{s,v}$); thus the propagated Wigner function is constant in time as it should be,

$$\rho_w(r, s; t) = \rho_w(r, s; t_0). \tag{61}$$

3.3. Comments on the mapped commutator

Surely, the mapped expression of the commutator deserves a deeper analysis. Although the sine term appears as in the continuous case, it must be observed that, due to the finite-dimensional character of the discrete phase space and the cyclic nature of the operators constituting the basis elements, the phase $\Phi(a, b, c, d; N)$ plays an essential role. In this form, equation (37) is the discrete phase space version of the continuous Moyal bracket, but in the discrete case it embodies all the peculiarities of the finite character of the space as well as the torus topology of the operator basis which are not present in the other case.

As mentioned before, the possibility of having a Schwinger basis for each prime N allows us to construct a basis for each one of the degrees of freedom associated with any finite-dimensional state space. Two extreme cases can be immediately separated and discussed from the infinitely many Ns. The first case is related to $N = 2$, the only even prime. For this case, it has been already argued that the Schwinger basis elements, as well as the one presented in [20], are directly related to the Pauli σ operators and obey well known commutation relations. Thus, they are directly associated with a spin-$\frac{1}{2}$ degree of freedom [10]. The second special case corresponds to assuming $N \to \infty$, when one expects to recover the canonical pair of operators, $\hat{Q}$ and $\hat{P}$, obeying the Heisenberg algebra and endowed with the standard commutation relation. This limit has been proposed by Schwinger and implemented directly in the unitary operators U and V [10], but the discussion of the behaviour of the *operator basis* in that limit is still lacking. Therefore, it seems to be of great interest to verify whether the operator bases proposed here, expression (22), gives back the Weyl–Wigner basis, and therefore the Moyal bracket, in that limit.

In the past, we have already shown that our operator bases correctly describe the commutation relations of the mapped operators associated with the time evolution of a magnetic dipole interacting with an external magnetic field (arbitrary finite N) [28]. In the opposite situation, the $N \to \infty$ limit case, must be related to the standard Weyl–Wigner formalism. In what follows we will show that our operator basis indeed gives back the Weyl–Wigner formalism in the $N \to \infty$ limit, and we will discuss the subtleties in obtaining the explicit limiting behaviour of the discrete mapped commutator.

4. Continuous limit of the DPSF

4.1. The Weyl–Wigner basis

The continuous limit of the discrete phase space formalism (DPSF) can be obtained as one takes the limit of the dimension of the state space, N, to infinity (through prime integers) in a prescribed form. To this end let us consider the proposed basis elements, equation (22),

$$\hat{G}(j, l) = \sum_{m,n=-h}^{h} \hat{T}(j, l; m, n).$$

Since the phase $\phi(m + h, n + h; N)$ will always be equal to zero in these sums, we may just write

$$\hat{G}(j, l) = \frac{1}{N} \sum_{m,n=-h}^{h} U^m V^n \exp\left[-\frac{2\pi \mathrm{i}}{N}(mj + nl)\right] \exp\left[\frac{\mathrm{i}\pi}{N} mn\right]. \qquad (62)$$

Let us also associate the operators

$$U = \exp[-\mathrm{i}\epsilon \hat{Q}] \qquad V = \exp[\mathrm{i}\epsilon \hat{P}] \qquad (63)$$

with

$$\epsilon^2 = \frac{2\pi}{N} \tag{64}$$

which becomes an infinitesimal as $N \to \infty$. We also perform the change of variables

$$-q = \epsilon\, j \qquad -p = \epsilon l \tag{65}$$

$$u = \epsilon m \qquad v = \epsilon n \tag{66}$$

defining the intervals

$$\Delta u = \epsilon\,\Delta m \qquad \Delta v = \epsilon\,\Delta n \tag{67}$$

with $\Delta m = \Delta n = 1$. These substitutions lead to

$$\hat{G}(p,q) = \frac{1}{\epsilon^2 N}\sum_{u,v=\epsilon h}^{-\epsilon h} \Delta u \Delta v \exp[-\mathrm{i}u\hat{Q}]\exp[+\mathrm{i}v\hat{P}]\exp[\mathrm{i}(qu+pv)]\exp\left[\tfrac{1}{2}\mathrm{i}uv\right]. \tag{68}$$

Considering $N \to \infty$ we have

$$\Delta u \to \mathrm{d}u \qquad \Delta v \to \mathrm{d}v$$

yielding

$$\hat{G}(p,q) = \frac{1}{2\pi}\int_{\infty}^{-\infty}\int_{\infty}^{-\infty} \mathrm{d}u\,\mathrm{d}v \exp[-\mathrm{i}u\hat{Q}]\exp[\mathrm{i}v\hat{P}]\exp\left[\tfrac{1}{2}\mathrm{i}uv\right]\exp[\mathrm{i}(qu+pv)] \tag{69}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}u\,\mathrm{d}v \exp[\mathrm{i}u(q+\tfrac{1}{2}v-\hat{Q})]\exp[\mathrm{i}v(p+\hat{P})] \tag{70}$$

which, with the help of the identity (that can be obtained through a similar limit of $|u_j\rangle\langle u_j| = \frac{1}{N}\sum_{k=-h}^{h} u_j^k U^k$),

$$|q\rangle\langle q| = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}x \exp[\mathrm{i}x(q-\hat{Q})] \tag{71}$$

gives

$$\hat{G}(p,q) = \int_{-\infty}^{\infty} \mathrm{d}v\, |q+\tfrac{1}{2}v\rangle\langle q+\tfrac{1}{2}v| \exp[\mathrm{i}v(p+\hat{P})] \tag{72}$$

$$\hat{G}(p,q) = \int_{-\infty}^{\infty} \mathrm{d}v \exp[\mathrm{i}vp]|q+\tfrac{1}{2}v\rangle\langle q-\tfrac{1}{2}v|. \tag{73}$$

Upon recalling equation (3), we identify

$$\hat{G}(p,q) = \Delta(p,q). \tag{74}$$

Once this result has been established, we can see that the expression giving the phase space mapping procedure can be rewritten as

$$\hat{A} = \frac{1}{N}\sum_{m,n=-h}^{h} \Delta m \Delta\, n A(m,n)\hat{G}(m,n) \tag{75}$$

such that, by assuming

$$p = \epsilon\, m \qquad q = \epsilon\, n$$

with ϵ as defined in expression (64), it follows that

$$\hat{A} = \frac{1}{2\pi} \sum_{m,n=-h}^{h} \Delta p \Delta q A(p,q) \hat{G}(p,q). \tag{76}$$

Upon taking the $N \to \infty$ limit we have

$$\hat{A} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}p \, \mathrm{d}q \; A(p,q) \Delta(p,q) \tag{77}$$

which is exactly equation (1) (up to an $\hbar$ factor). Once the basis elements are reattained, all further Weyl–Wigner results follow directly, and, in fact, they can also be obtained from their discrete phase space counterparts just by taking the continuous limit in the same form as we have done above.

4.2. The commutator and the Moyal bracket

In the previous sections we have emphasized the role played by the modular phase ϕ in connection with the boundary conditions. Due to its importance, we will perform in detail the $N \to \infty$ limit for the product of two operators, which will lead us back to the continuous Moyal bracket. To this end let us start from equation (33) and use expression (39),

$$\begin{aligned} (\hat{O}_1 \hat{O}_2)(m,n) = \frac{1}{N^4} \sum_{u,v,r,s=-h}^{h} \; \sum_{a,b,c,d=-h}^{h} \Delta u \Delta v \Delta r \Delta s \Delta a \Delta b \Delta c \Delta d \\ \times O_1(u,v) O_2(r,s) \exp\left[\frac{\mathrm{i}\pi}{N}(bc - ad)\right] \exp[\mathrm{i}\pi \Phi(a,b,c,d;N)] \\ \times \exp\left[\frac{2\pi \mathrm{i}}{N}[a(m-u) + b(n-v) + c(m-r) + d(n-s)]\right]. \end{aligned} \tag{78}$$

If we perform the change of variables

$$\begin{aligned} a + c = j \qquad b + d = l \\ a - c = x \qquad b - d = z \end{aligned} \tag{79}$$

the mapping of the product then reads,

$$\begin{aligned} (\hat{O}_1 \hat{O}_2)(m,n) = \frac{1}{N^4} \sum_{u,v,r,s=-h}^{h} \; \sum_{x=-(2h-|j|)}^{2h-|j|} \; \sum_{z=-(2h-|l|)}^{2h-|l|} \; \sum_{j,l=-2h}^{2h} \Delta u \Delta v \Delta r \Delta s \Delta j \Delta l \tfrac{1}{4} \Delta x \Delta z \\ \times O_1(u,v) O_2(r,s) \exp\left[\frac{\mathrm{i}\pi}{2N}(jz - lx)\right] \exp\left[\mathrm{i}\pi \phi(j+h, l+h; N)\right] \\ \times \exp\left\{\frac{\pi \mathrm{i}}{N}\left[j\,(2m - r - u) + x\,(r - u) + l\,(2n - s - v) + z\,(s - v)\right]\right\} \end{aligned} \tag{80}$$

where summations over the indices $\{x, z\}$ are restricted to run in steps of 2. Realizing that the phase ϕ has a discontinuous nature, we see that it is convenient to break the summations over $\{j, l\}$ into nine different intervals, according to the different values assumed by the phase, namely

$$\begin{aligned} \sum_{j,l=-2h}^{2h} = \sum_{j,l=-h}^{h} + \sum_{j,l=-2h}^{-(h+1)} + \sum_{j=-2h}^{-(h+1)} \sum_{l=-h}^{h} + \sum_{j=-2h}^{-(h+1)} \sum_{l=h+1}^{2h} \\ + \sum_{j=-h}^{h} \sum_{l=-2h}^{-(h+1)} + \sum_{j=-h}^{h} \sum_{l=h+1}^{2h} + \sum_{j=h+1}^{2h} \sum_{l=-2h}^{-(h+1)} + \sum_{j=h+1}^{2h} \sum_{l=-h}^{h} + \sum_{j,l=h+1}^{2h}. \end{aligned} \tag{81}$$

Only the first of these nine different summations will give a non-zero contribution in the continuous limit, which we will calculate in detail. On the other hand, just one of the remaining eight terms will be calculated, since all others vanish in a similar fashion.

Let us consider the first sum in equation (81) contributing to equation (80), which we denote by $(\hat{O}_1\hat{O}_2)_1(m,n)$, i.e. the non-zero contribution. As $\{j,l\}$ both lie in the interval $[-h,h]$, the phase ϕ is equal to zero. We consider again equation (64) with the introduction of the new variables

$$\epsilon\, m = p \qquad \epsilon\, n = q \tag{82}$$

and, for the remaining labels, we adopt

$$\epsilon\, y = \bar{y}. \tag{83}$$

The summations are once again replaced by integrals as the scaled variables $\bar{y}$ become continuous when $N \to \infty$ ($\epsilon \to 0$). Then, since the integration is performed over all space, the constraints in the $\{\bar{x},\bar{z}\}$ summing intervals can be dropped, so that

$$\begin{aligned}(\hat{O}_1\hat{O}_2)_1(p,q) = {}& \frac{1}{4\epsilon^8 N^4}\int_{-\infty}^{\infty} \mathrm{d}\bar{u}\,\mathrm{d}\bar{v}\,\mathrm{d}\bar{r}\,\mathrm{d}\bar{s}\,\mathrm{d}\bar{j}\,\mathrm{d}\bar{l}\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{z}\; O_1(\bar{u},\bar{v})O_2(\bar{r},\bar{s}) \\ & \times \exp\left\{\tfrac{1}{4}\mathrm{i}\bar{z}\left[\bar{j} - 2\,(v-\bar{s})\right]\right\}\exp\left\{\tfrac{1}{4}\mathrm{i}x\left[\bar{l} - 2\,(\bar{r}-\bar{u})\right]\right\} \\ & \times \exp\left\{\tfrac{1}{2}\mathrm{i}\left[\bar{j}\,(2m-\bar{r}-\bar{u}) + l\,(2n-\bar{s}-\bar{v})\right]\right\}.\end{aligned} \tag{84}$$

Integration over $\{\frac{1}{4}\bar{x},\frac{1}{4}\bar{z}\}$ yields $\delta[\bar{l}-2\,(\bar{r}-\bar{u})]$ and $\delta[j-2\,(\bar{v}-\bar{s})]$, respectively, in such a form that the integration over $\{\bar{j},\bar{l}\}$ gives

$$\begin{aligned}(\hat{O}_1\hat{O}_2)_1(p,q) = {}& \frac{4}{(2\pi)^2}\int \mathrm{d}\bar{u}\,\mathrm{d}\bar{v}\,\mathrm{d}\bar{r}\,\mathrm{d}\bar{s}\; O_1(\bar{u},\bar{v})O_2(\bar{r},\bar{s}) \\ & \times \exp\left\{\tfrac{1}{2}\mathrm{i}\left[(\bar{v}-\bar{s})(2p-(\bar{u}+\bar{r})) + (\bar{r}-\bar{u})(2q-(\bar{v}+\bar{s}))\right]\right\}\end{aligned} \tag{85}$$

that can be finally written as

$$\begin{aligned}(\hat{O}_1\hat{O}_2)_1(p,q) = {}& \frac{4}{(2\pi)^2}\int \mathrm{d}\bar{u}\,\mathrm{d}\bar{v}\,\mathrm{d}\bar{r}\,\mathrm{d}\bar{s}\; O_1(\bar{u},\bar{v})O_2(\bar{r},\bar{s}) \\ & \times \exp\left\{2\mathrm{i}\left[(\bar{s}-q)(\bar{u}-p) - (\bar{v}-q)(\bar{r}-p)\right]\right\}\end{aligned} \tag{86}$$

which is exactly (up to an $\hbar$ factor) equation (10). From this expression one can straightforwardly obtain the Moyal bracket.

All the remaining sums can be treated in exactly the same way as above, giving obviously analogous expressions except for the phase ϕ, which will now be different from zero. This will give rise to an additional phase with respect to the previous result. For example, considering the contribution

$$\begin{aligned}(\hat{O}_1\hat{O}_2)_9(m,n) = {}& \frac{1}{N^4}\sum_{u,v,r,s=-h}^{h}\;\sum_{x=-(2h-|j|)}^{2h-|j|}\;\sum_{z=-(2h-|l|)}^{2h-|l|}\;\sum_{j,l=h+1}^{2h} \Delta u\,\Delta v\,\Delta r\,\Delta s\,\Delta j\,\Delta l\,\tfrac{1}{4}\Delta x\,\Delta z \\ & \times O_1(u,v)O_2(r,s)\exp\left[\frac{\mathrm{i}\pi}{2N}(jz-lx)\right]\exp\left[\mathrm{i}\pi\phi(j+h,l+h;N)\right] \\ & \times \exp\left\{\frac{\pi\mathrm{i}}{N}\left[j\,(2m-r-u) + x\,(r-u) + l\,(2n-s-v) + z\,(s-v)\right]\right\}\end{aligned} \tag{87}$$

it is easy to see that the presence of the term $\exp[i\pi(j+l+N)]$ changes the behaviour of this expression, respectively, to what has been done before. To evaluate it one should, before scaling the variables from $\{j,l\}$ to $\{\bar{j},\bar{l}\}$, break up these summations into odd and even summing labels. So, let us rewrite the above expression as

$$(\hat{O}_1\hat{O}_2)_9(m,n) = \frac{1}{N^4}\sum_{u,v,r,s=-h}^{h}\sum_{x=-(2h-|j|)}^{2h-|j|}\sum_{z=-(2h-|l|)}^{2h-|l|}\left(\sum_{j,l=h+1}^{2h}\right)\Delta u\Delta v\Delta r\Delta s\Delta j\Delta l\tfrac{1}{4}\Delta x\Delta z$$
$$\times O_1(u,v)O_2(r,s)\exp\left[\frac{i\pi}{2N}(jz-lx)\right]\exp\left[i\pi\phi(j+h,l+h;N)\right]$$
$$\times\exp\left\{\frac{\pi i}{N}\left[j\,(2m-r-u)+x\,(r-u)+l\,(2n-s-v)+z\,(s-v)\right]\right\} \qquad (88)$$

where we have used the shortened form

$$\left(\sum_{j,l=h+1}^{2h}\right) = \sum_{j=h+1,even}^{2h}\sum_{l=h+1,even}^{2h} + \sum_{j=h+1,even}^{2h}\sum_{l=h+1,odd}^{2h} + \sum_{j=h+1,odd}^{2h}\sum_{l=h+1,even}^{2h}$$
$$+\sum_{j=h+1,odd}^{2h}\sum_{l=h+1,odd}^{2h}. \qquad (89)$$

Since the exponential $\exp[i\pi(j+l+N)]$ depends only on the parity of the labels $\{j,l\}$, its contribution to the summations can be evaluated *a priori* yielding +1 for the first and last term of the right-hand side of equation (89) and -1 for the other two. As the labels j and l are scaled and we take the continuous limit, there is no meaning in keeping the even/odd character of the labels, and the sums will cancel out pairwise. The same reasoning is valid for all similar terms in equation (81). That can also be seen in an intuitive fashion, realizing that in the continuous limit the term $\exp[i\pi(j+l+N)]$ will oscillate infinitely rapidly, and the subsequent integration naturally vanishes.

It is important to stress that this only happens in the continuous infinite-dimensional limit. In finite-N cases these sums do not vanish. We may also study the effects of this cancellation on the expression for the commutator and the related Moyal bracket. In this connection, it can be immediately seen that the expression for the continuous Moyal bracket casts its roots in only one term of the discrete expression for the commutator since all others do not contribute in that limit. Besides, in the limit process, some sums, that *were not* representations of Kronecker deltas in the discrete case, can be performed as they were transformed into integrals, equations (84) and (85), yielding Dirac deltas which, in turn, were immediately integrated.

Therefore, in the opposite sense, we realize that one cannot recover the discrete commutator by simply discretizing the continuous Moyal bracket, first because the summations just mentioned cannot be recovered in a univocal way from the continuous result, and second, according to equation (86), the Moyal bracket is associated with only one contribution of the discrete dynamical bracket. In fact, one can only recover the discrete commutator from the continuous one, through the introduction of the phases and summations discussed above in a very artificial way. This clearly indicates that simply discretized Moyal bracket schemes intended for the description of the dynamics of finite-dimensional degrees of freedom are likely to fail.

5. Concluding remarks

In the previous sections we first recalled the Weyl–Wigner formalism which allows for a continuous phase space description of quantum systems. It is well known that such a description is suitable for degrees of freedom described by continuous variables, although it fails for degrees of freedom described by a finite-dimensional state space. We have then pointed out that a formalism has been developed in the past that accounts for such systems when discrete quantum phase spaces can then be constructed, which are labelled by discrete variables, that can also be used to describe degrees of freedom without classical counterparts. The first outcome of our work is that we have unified the approach for the discrete and continuous cases.

One important feature of the discrete phase space formalism is that, due to a factorization property of operator bases, as noticed by Schwinger, we can write a mapping kernel which maps operators onto functions of discrete variables in the discrete phase space, for each one of the degrees of freedom. Obviously, we can consider $N \to \infty$ as a particular and interesting case, since we expect it to agree with the Weyl–Wigner description. We also stress that the symmetrization introduced here has ensured that our new basis elements are Hermitian, which has led to a simple expression for the trace of the two operators, equation (36) (which has direct and very important consequences on the calculation of expected values), and also has ensured the limit to the Weyl–Wigner case.

Our first result consists in, as mentioned above, starting from a general expression for the discrete mapping kernel, to show that the Weyl–Wigner continuous mapping kernel is indeed reattained as a limiting case of the discrete one. Furthermore, one important result also emerged. Let us recall that when we consider the dynamics in a continuous phase space we are directly led to the Moyal bracket, which is the quantum analogue of the Poisson bracket. On the other hand, we have a discrete bracket that governs the dynamics in discrete finite-dimensional phase spaces. This discrete dynamical bracket, besides being endowed with a presymplectic structure [23], has also being shown to properly evolve discrete Wigner functions in time through a discrete finite-dimensional phase space [28]. Now, the discrete mapping kernels are modulo N invariant as required [13], and this is accomplished through the presence of phases dealing with integral parts of integers. When discussing the $N \to \infty$ limit of the discrete dynamical bracket, we notice a vanishing—due precisely to the subtle and essential role played by the referred phases and the symmetrization of the interval of the phase space labels—of all but one term which then allows the final limiting expression to coincide with the Moyal bracket. As a result of this, it becomes clear that the Moyal bracket, or discretized versions of it, are unable to correctly describe the dynamics of genuine finite-dimensional degrees of freedom. A further remark is that, since the Poisson bracket is known to emerge from the Moyal bracket, here shown to be a limiting case of a discrete bracket, it is interesting to observe that the classical result casts its roots in the mathematical formulation which is suitable for the description of degrees of freedom without classical counterparts.

As a simple and straightforward conclusion, we state that the present method can be proposed as a unified way of treating dynamics in quantum phase spaces, either continuous or discrete finite dimensional.

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Appendix. Trace of three basis elements

Starting from the definition,

$$G(m,n)=\frac{1}{N}\sum_{j,l=-h}^{h}U^{j}V^{l}\exp\left[\frac{\pi\mathrm{i}}{N}jl\right]\exp\left[\mathrm{i}\pi\phi(j,l;N)\right]\exp\left[-\frac{2\pi\mathrm{i}}{N}(mj+nl)\right] \tag{A1}$$

we write explicitly

$$\begin{aligned}\mathrm{Tr}[G^{\dagger}(m,n)\,G(u,v)\,G(r,s)]\\ &=\sum_{k=0}^{N-1}\langle v_k|\frac{1}{N}\sum_{j,l=-h}^{h}U^{-j}V^{-l}\exp\left[\frac{\pi\mathrm{i}}{N}jl\right]\exp\left[-\mathrm{i}\pi\phi(j+h,l+h;N)\right]\\ &\quad\times\exp\left[\frac{2\pi\mathrm{i}}{N}(mj+nl)\right]\frac{1}{N}\sum_{a,b=-h}^{h}U^{a}V^{b}\exp\left[\frac{\pi\mathrm{i}}{N}ab\right]\\ &\quad\times\exp\left[\mathrm{i}\pi\phi(a+h,b+h;N)\right]\exp\left[-\frac{2\pi\mathrm{i}}{N}(ua+bv)\right]\frac{1}{N}\sum_{c,d=-h}^{h}U^{c}V^{d}\\ &\quad\times\exp\left[\frac{\pi\mathrm{i}}{N}cd\right]\exp\left[\mathrm{i}\pi\phi(c+h,d+h;N)\right]\exp\left[-\frac{2\pi\mathrm{i}}{N}(cr+\mathrm{d}s)\right]|v_{k\rangle}\end{aligned} \tag{A2}$$

$$\begin{aligned}\mathrm{Tr}[G^{\dagger}(m,n)\,G(u,v)\,G(r,s)]&=\frac{1}{N^3}\sum_{k=0}^{N-1}\sum_{j,l=-h}^{h}\sum_{a,b,c,d=-h}^{h}\langle v_k|U^{-j}V^{-l}U^{a}V^{b}U^{c}V^{d}|v_k\rangle\\ &\quad\times\exp\left[-\mathrm{i}\pi(\phi(j+h,l+h;N)\right]\exp\left[\mathrm{i}\pi\phi(a+h,b+h;N)\right]\\ &\quad\times\exp\left[\mathrm{i}\pi\phi(c+h,d+h;N)\right]\exp\left[\frac{\pi\mathrm{i}}{N}(jl+ab+cd)\right]\\ &\quad\times\exp\left[\frac{2\pi\mathrm{i}}{N}(mj+nl-ua-bv-cr-ds)\right].\end{aligned} \tag{A3}$$

Now let us calculate in detail the matrix element in (A3) using the Weyl–Schwinger commutation relation $V^{x}U^{y}=U^{y}V^{x}\exp((2\pi\mathrm{i}/N)xy)$,

$$\langle v_k|U^{-j}V^{-l}U^{a}V^{b}U^{c}V^{d}|v_k\rangle=\langle v_k|U^{a+c-j}V^{b+d-l}|v_k\rangle\exp\left[\frac{2\pi\mathrm{i}}{N}(cb-cl-al)\right] \tag{A4}$$

$$\begin{aligned}\langle v_k|U^{-j}V^{-l}U^{a}V^{b}U^{c}V^{d}|v_k\rangle&=\langle v_k|v_{k+a+c-j}\rangle\exp\left[\frac{2\pi\mathrm{i}}{N}k(b+d-l)\right]\\ &\quad\times\exp\left[\frac{2\pi\mathrm{i}}{N}(cb-cl-al)\right]\end{aligned} \tag{A5}$$

$$\langle v_k|U^{-j}V^{-l}U^{a}V^{b}U^{c}V^{d}|v_k\rangle=\delta^{[N]}_{a+c-j,0}\exp\left[\frac{2\pi\mathrm{i}}{N}k(b+d-l)\right]\exp\left[\frac{2\pi\mathrm{i}}{N}(cb-cl-al)\right]. \tag{A6}$$

Rewriting (A3),

$$\mathrm{Tr}[G^{\dagger}(m,n)\,G(u,v)\,G(r,s)] = \frac{1}{N^3}\sum_{k=0}^{N-1}\sum_{j,l=-h}^{h}\sum_{a,b,c,d=-h}^{h}\exp\left[\frac{2\pi\mathrm{i}}{N}k(b+d-l)\right]\delta^{[N]}_{a+c-j,0}$$
$$\times\exp[-\mathrm{i}\pi(\phi(j+h,l+h;N)]\exp[\mathrm{i}\pi\phi(a+h,b+h;N)]$$
$$\times\exp[\mathrm{i}\pi\phi(c+h,d+h;N)]\exp\left[\frac{2\pi\mathrm{i}}{N}(cb-cl-al)\right]$$
$$\times\exp\left[\frac{\pi\mathrm{i}}{N}(+jl+ab+cd)\right]\exp\left[\frac{2\pi\mathrm{i}}{N}(mj+nl-ua-bv-cr-ds)\right] \tag{A7}$$

the summation over the $\{k\}$ label yields a delta function

$$\mathrm{Tr}[G^{\dagger}(m,n)\,G(u,v)\,G(r,s)] = \frac{1}{N^2}\sum_{j,l=h}^{h}\sum_{a,b,c,d=-h}^{h}\delta^{[N]}_{a+c-j,0}\delta^{[N]}_{b+d-l,0}$$
$$\times\exp[-\mathrm{i}\pi(\phi(j+h,l+h;N)]\exp[\mathrm{i}\pi\phi(a+h,b+h;N)]$$
$$\times\exp[\mathrm{i}\pi\phi(c+h,d+h;N)]\exp\left[\frac{2\pi\mathrm{i}}{N}(cb-l(c+a)\right]$$
$$\times\exp\left[\frac{\pi\mathrm{i}}{N}(jl+ab+cd)\right]\exp\left[\frac{2\pi\mathrm{i}}{N}(mj+nl-ua-bv-cr-ds)\right]. \tag{A8}$$

Summing over $\{j,l\}$, and making use of equation (26), we obtain

$$\mathrm{Tr}[G^{\dagger}(m,n)\,G(u,v)\,G(r,s)] = \frac{1}{N^2}\sum_{a,b,c,d=-h}^{h}\exp\left[\frac{2\pi\mathrm{i}}{N}(cb-(b+d)(a+c))\right]$$
$$\times\exp[-\mathrm{i}\pi(\phi(a+c+h,b+d+h;N)]\exp[\mathrm{i}\pi\phi(a+h,b+h;N)]$$
$$\times\exp[\mathrm{i}\pi\phi(c+h,d+h;N)]\exp\left[\frac{\pi\mathrm{i}}{N}((a+c)(b+d)+ab+cd)\right]$$
$$\times\exp\left[\frac{2\pi\mathrm{i}}{N}(m(a+c)+n(b+d)-ua-bv-cr-ds)\right]. \tag{A9}$$

Reordering and recalling that $\exp[\mathrm{i}\pi\phi(a+h,b+h;N)] = \exp[\mathrm{i}\pi\phi(c+h,d+h;N)] = 1$ in the considered ranges,

$$\mathrm{Tr}[G^{\dagger}(m,n)\,G(u,v)\,G(r,s)] = \frac{1}{N^2}\sum_{a,b,c,d=-h}^{h}\exp[-\mathrm{i}\pi\phi(a+c+h,b+d+h;N)]$$
$$\times\exp\left[\frac{\pi\mathrm{i}}{N}(bc-ad)\right]$$
$$\times\exp\left[-\frac{2\pi\mathrm{i}}{N}\,(a(m-u)+b(n-v)+c(m-r)+d(n-s))\right]. \tag{A10}$$

References

[1] Weyl H 1978 *The Theory of Groups and Quantum Mechanics* (New York: Dover)
[2] Wigner E 1932 *Phys. Rev.* **40** 749
[3] Moyal J E 1949 *Proc. Camb. Phil. Soc.* **45** 99

[4] Leaf B 1968 *J. Math. Phys.* **9** 65
[5] Hillery M, O'Connel R F, Scully M O and Wigner E P 1984 *Phys. Rep.* C **106** 347
[6] Balazs N L and Jennings B K 1984 *Phys. Rep.* C **104** 347
[7] Ozório de Almeida A M 1998 *Phys. Rep.* **295** 265
[8] Kim Y S and Noz M E 1991 *Phase Space Picture of Quantum Mechanics (Lecture Notes in Physics vol 40)* (Singapore: World Scientific)
[9] Gelfand I M and Fairlie D B 1991 *Commun. Math. Phys.* **136** 487
[10] Schwinger J 1960 *Proc. Nat. Acad. Sci., USA* **46** 570
Schwinger J 1960 *Proc. Nat. Acad. Sci., USA* **46** 893
Schwinger J 1960 *Proc. Nat. Acad. Sci., USA* **46** 1401
Schwinger J 1961 *Proc. Nat. Acad. Sci., USA* **47** 1075
[11] Fairlie D, Fletcher P and Zachos C 1990 *J. Math. Phys.* **31** 1088
[12] Hakioğlu T 1998 *J. Phys. A: Math. Gen.* **31** 6975
[13] Galetti D and de Toledo Piza A F R 1988 *Physica* A **149** 267
[14] Zak J 1989 *J. Math. Phys.* **30** 1591
[15] Zak J 1989 *J. Math. Phys.* **39** 694
[16] Cohendet O, Combe Ph, Siruge M and Siruge-Collin M 1988 *J. Phys. A: Math. Gen.* **21** 2875
[17] Wooters W K 1987 *Ann. Phys.* **176** 1
[18] Ellinas D and Floratos E G 1999 *J. Phys. A: Math. Gen.* **32** L63
[19] Digernes T, Varadarajan V S and Varadhan S R S 1994 *Rev. Math. Phys.* **6** 621
[20] Galetti D and de Toledo Piza A F R 1992 *Physica* A **186** 513
[21] Galetti D and de Toledo Piza A F R 1995 *Physica* A **214** 207
[22] Galetti D and Marchiolli M A 1996 *Ann. Phys.* **249** 454
[23] Aldrovandi R and Galetti D 1990 *J. Math. Phys.* **31** 2987
[24] de Groot S R and Suttorp L G 1972 *Foundations of Electrodynamics* (Amsterdam: North-Holland)
[25] Baker G 1958 *Phys. Rev.* **109** 2198
[26] Bayen F, Flato M, Lichnerowicz A and Sternheimer D 1978 *Ann. Phys., NY* **111** 61
Bayen F, Flato M, Lichnerowicz A and Sternheimer D 1978 *Ann. Phys., NY* **111** 111
[27] Curtwright T, Fairlie D and Zachos C 1998 *Phys. Rev.* D **58** 025002-1
[28] Galetti D and Ruzzi M 1999 *Physica* A **264** 473